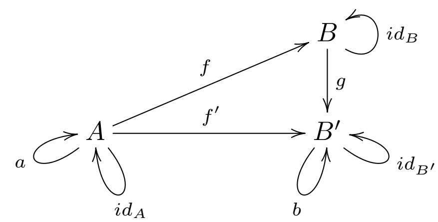
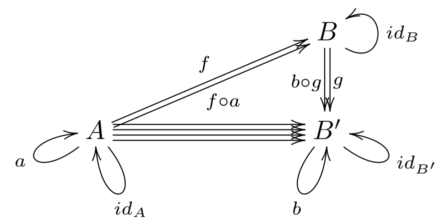
# 1 Covariant Hom functors

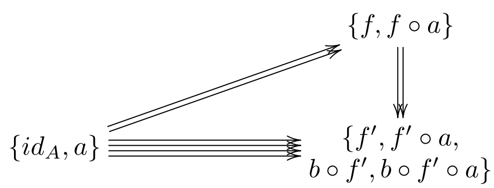
These are defined in §3.20.4, but a longer example never hurt anybody, right?
Suppose $a^2 = id_A$, $b^2 = id_{B'}$, and $f' = g \circ f$ in this category **A**.




If we actually draw out all the arrows, we get this diagram:




where the four arrows from $A$ to $B'$ are $\{f', f' \circ a, b \circ f', b \circ f' \circ a\}$.
$\hom(A, -)$ is the following full subcategory of **Set**; please note the similarities—the *representation* of the structure reachable from $A$:




The two arrows from $\hom(A, A) = \{id_A, a\}$ to $\hom(A, B) = \{f, f \circ a\}$ are obtained by post-composition:

$$\hom(A, f) = \{id_A \mapsto f, a \mapsto f \circ a\}$$
$$\hom(A, f \circ a) = \{id_a \mapsto f \circ a, a \mapsto f\}$$

(the last entry holds because $(f \circ a) \circ a = f \circ (a \circ a) = f \circ id_A = f$). The four arrows from $\hom(A, A)$ to $\hom(A, B') = \{f', f' \circ a, b \circ f', b \circ f' \circ a\}$ are again obtained by post-composition:

$$\hom(A, f') = \{id_A \mapsto f', a \mapsto f' \circ a\}$$
$$\hom(A, f' \circ a) = \{id_a \mapsto f' \circ a, a \mapsto f'\}$$
$$\hom(A, b \circ f') = \{id_a \mapsto b \circ f', a \mapsto b \circ f' \circ a\}$$
$$\hom(A, b \circ f' \circ a) = \{id_a \mapsto b \circ f' \circ a, a \mapsto b \circ f'\}$$

The two vertical arrows are (again by post-composition, and recall that $f' = g \circ f$):

$$\hom(A, g) = \{f \mapsto f', f \circ a \mapsto f' \circ a\}$$
$$\hom(A, b \circ g) = \{f \mapsto b \circ f', f \circ a \mapsto b \circ f' \circ a\}$$

It is easy to check that, indeed, composition still holds: the four horizontal arrows are each the result of composition of a choice of vertical and diagonal arrows, and we haven't missed any.

## 2 Proposition 6.18 and The Yoneda Lemma

Let's restrict our attention to this category **A** (to truly appreciate the significance of this result, I encourage you to work out the details in full for a slightly larger category!):

$$A \xrightarrow{f} B$$

The image of this in **Set** under $\hom(A, -)$ is just

$$\{id_A\} \xrightarrow{\hom(A,f)} \{f\}$$

Now suppose we have some other functor $F : \mathbf{A} \to \mathbf{Set}$, whose image is

$$\{a_0, \dots\} \xrightarrow{Ff} \{b_0, \dots\}$$

(where $FA = \{a_0, \dots\}$ and $FB = \{b_0, \dots\}$.)

Now, the claim of Proposition 6.18 is that there exists a unique natural transformation $\tau : \hom(A, -) \stackrel{\cdot}{\to} F$ if we additionally constrain $\tau_A(id_A) = a_0$. OK, so, first off: what does that mean? $\tau$ being natural means $\forall B, C, g : B \to C$, this commutes:

$$\begin{array}{ccc} \hom(A,B) & \xrightarrow{\tau_B} & FB \\ {\scriptstyle \hom(A,g)} \downarrow & & \downarrow {\scriptstyle Fg} \\ \hom(A,C) & \xrightarrow{\tau_C} & FC \end{array}$$

or more specifically, at $A, B, f$ (first generically, then expanding some computations):

$$\begin{array}{ccc} \hom(A,A) & \xrightarrow{\tau_A} & FA \\ {\scriptstyle \hom(A,f)} \downarrow & & \downarrow {\scriptstyle Ff} \\ \hom(A,B) & \xrightarrow{\tau_B} & FB \end{array} \qquad \begin{array}{ccc} \{id_A\} & \xrightarrow{\tau_A} & \{a_0, \dots\} \\ {\scriptstyle \{id_A \mapsto f\}} \downarrow & & \downarrow {\scriptstyle Ff} \\ \{f\} & \xrightarrow{\tau_B} & \{b_0, \dots\} \end{array}$$

and so requiring $\tau_A(id_A) = a_0$ makes sense. If this is to be natural, it must be the case (for all $B$ and $f : A \to B$; note that this works even to define $\tau_A$ at inputs other than $id_A$ just as well!) that

$$\begin{aligned} \tau_B(f) &= \tau_B(f \circ id_A) & \\ &= \tau_B(\hom(A,f)(id_A)) & \forall_x . f \circ x = \hom(A,f)(x) \\ &= F(f)(\tau_A(id_A)) & \text{naturality of } \tau \\ &= F(f)(a_0) & \text{requirement} \end{aligned}$$

So $\tau$ is fully determined by naturality and the requirement given, precisely because $\hom(A, -)$ on arrows captures post-composition. So: given a choice of $a_0 \in FA$, we can fully specify a natural transformation $\tau$.

Conversely, given a $\tau'$, it must pick out some $\tau_A(id_A) \in FA$. Therefore, the Yoneda lemma:

> *Given a functor $F : \mathbf{A} \to \mathbf{Set}$, the set $\left\{\tau \middle| \tau : hom(A, -) \stackrel{\cdot}{\to} F\right\}$ is isomorphic (in **Set**) to $FA$. The isomorphism is witnessed by the function $Y(\tau) = \tau_A(id_A)$.*