1 Intro

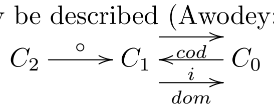
Unless otherwise notated, references are to *Abstract and Concrete Categories: The Joy of Cats*, [JHS04]. Notation follows theirs with some contamination from Awodey's *Category Theory* [Awo10], Pierce's *Basic Category Theory for Computer Scientists* [Pie91], and Riehl's *Category Theory in Context* [Rie].
Entries within each section are roughly sorted by definition, alphabetically.
Quantifiers are written perhaps unusually in this document, as Q_ϕ, where Q is $\forall$, $\exists$, $\bigcup$, etc. and ϕ is a list of variables or an expression whose free variables are quantified over. Constrained quantification may be written as $v_1 : \tau_1, v_2 : \tau_2.\phi(v_1, v_2)$ to indicate "the pairs of values v_1 ($\in \tau_1$) and v_2 ($\in \tau_2$) such that $\phi(v_1, v_2)$ holds". Strings of quantifiers are represented $Q_\phi Q'_{\phi'}$ etc. There is not necessarily a dot between quantifiers or between the quantifiers and quantified formula.

2 Basics

¶1 A **category C** (§3.1) is a quadruple $(\mathcal{O}, \mathrm{Hom}, id, \circ)$ with

- A collection of objects $\mathcal{O}$
- For each pair of objects A, B, a (disjoint) collection of arrows from **domain** A to **codomain** B, $\mathrm{Hom}(A, B)$ (also written $\mathbf{C}(A, B)$).
- An associative arrow composition operator $\circ$.
- Identity arrows (id_A) on each object A, unit of $\circ$

¶2 Categories may be described (Awodey:p21) as



¶3 A category is (Awodey:p24-25,D1.11-12)...

- **small** if C_0 and C_1 are sets and **large** otherwise.
- **locally small** if $\forall_{X,Y \in C_0} \mathrm{Hom}_C(X, Y) \subseteq C_1$ is a set.

¶4 A predicate P is **essentially unique** (§7.3) if it is unique up to isomorphism:

- If both PA and PB, then $A \simeq B$
- If PA and $A \simeq B$, then PB.

¶5 **B** is a **subcategory** of **A** if it has subcollections of objects and morphisms with identical composition and identity (§4.1.1). **B** is additionally ...

- **full** if it has all morphisms from **A** between objects in **B**. (§4.1.2)
- **reflective** if each B has an **A**-reflection. (§4.16.2) ¶26

¶6 A category is...

- **balanced** if all bi are iso (§7.49.2)
- **discrete** if all morphisms are identities. (§3.26.1)
- **thin** if $\forall_{A,B} \mathrm{Hom}(A, B) \simeq \{*\}$. (§3.26.2)

3 Derived Categories

¶7 The **arrow** (Awodey:p16,i3) category $\mathbf{C}^{\rightarrow}$ has arrows for commutative squares in **C**. There are two functors $\mathbf{cod}, \mathbf{dom} : \mathbf{C}^{\rightarrow} \rightarrow \mathbf{C}$.

¶8 The **cone** category over a given diagram, $\mathbf{Cone}(D(J))$, has as objects cones[¶57] to that diagram and a morphism between cones is an arrow $\phi : C \rightarrow C'$ s.t. $\forall_{D_j \in D(J)} c'_j \circ \phi = c_j$.

¶9 The **dual** (§3.5;Awodey:p15,i2) category $\mathbf{A}^{\mathrm{op}}$ which exchanges domains and codomains of arrows in **A**. Any purely-categorical statement implies its dual.

¶10 The **slice** (Awodey:p16,i4) category $\mathbf{C}/C$ has objects of arrows in **C** with codomain C. Arrows are tops of commutative triangles.

4 Object Properties

¶11 C is a **coseparator** if $\forall_{f,g:B\rightarrow A} f \neq g \Rightarrow \exists_{h:A\rightarrow C}.h \circ f \neq h \circ g$. (§7.17) (Contrast monomorphism[¶28].)

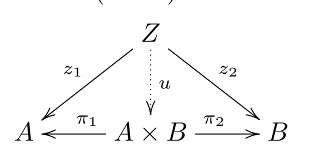
¶12 An object 0 is **initial** if $\forall_B \exists!_{f_B:0\rightarrow B} \top$. (§7.1)

¶13 A **limit** (Awodey:D5.16) of a diagram $D(J)$ is a terminal object in the category $\mathbf{Cone}(D(J))$. Written: $c_i : (\varprojlim_j D_j) \rightarrow D_i$. A **colimit** (Awodey:§5.6) is an initial object in the category of cocones; $c_i : D_i \rightarrow (\varinjlim_j D_j)$. ¶57

¶14 The pair of $(\Pi A.X) \in \mathbf{C}$ and $\pi : A \rightarrow \mathrm{Hom}_{\mathbf{C}}(\Pi A.X, X)$ is the **power** of A : **Set** and X : **C** if $\forall_{B:\mathbf{C},g:A\rightarrow \mathrm{Hom}_{\mathbf{C}}(B,X)} \exists_{(\Delta_{a\in A}.g(a))\in \mathrm{Hom}_{\mathbf{C}}(B,\Pi A.X)} \forall_{f\in \mathrm{Hom}_{\mathbf{C}}(B,\Pi A.X)}.f = \Delta_{a\in A}.g(a) \Leftrightarrow \lambda_{\hat{a}\in A}.\pi(\hat{a}) \cdot f = g$. (see [Hin]).

¶15 $(A \times B, \pi_1, \pi_2)$ is a **product** iff (UMP)

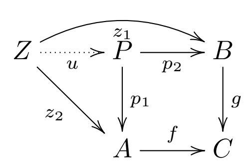
$$\forall_{Z,z_1,z_2} \exists!_u \quad u\pi_1 = z_1 \ \wedge \ u\pi_2 = z_2$$



¶16 The **product category C × D** of two categories **C** and **D** consists of objects which are each an ordered pair of an object from **C** and one from **D**; morphisms are, similarly, pairs of morphisms from **C** and **D**. This sense of × is itself the trivial bifunctor[¶45].

¶17 (P, p_1, p_2) is a **pullback** (Awodey:p80,D5.4) of f, g iff (UMP)

$$\forall_{Z,z_1,z_2.fz_1=gz_2} \exists!_u \quad z_1 = p_1u \ \wedge \ z_2 = p_2u$$



P may be denoted $A \times_C B$ when f, g are clear.

¶18 S is a **separator** if $\forall_{f,g:A\to B} f \neq g \Rightarrow \exists_{h:S\to A}. f\circ h \neq g \circ h$. (§7.10) (Contrast epimorphism[¶23].) S is a separator iff $\mathrm{Hom}(S,-)$ is faithful. (§7.12)

¶19 A set of objects $\mathcal{T}$ is a **separating set** if $\forall_{f,g:A\to B} f \neq g \Rightarrow \exists S \in \mathcal{T}, h : S \to A. f \circ h \neq g \circ h$. (§7.14)

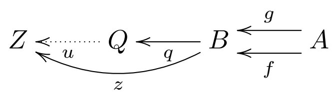
¶20 An object 1 is **terminal** if $\forall_A \exists!_{f:A\to 1}\top$. (§7.4)

¶21 An object that is both initial and terminal is called a **zero**. (§7.7) EX: ¶85

5 Arrow Properties

¶22 (Q, q) is a **coequalizer** (§7.51) of f, g iff (UMP) $qf = qg$ and

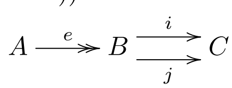
$\forall_{Z,z.zf=zg}\exists!_u uq = z$



Coequalizers are essentially unique (§7.70.1) and epic (§7.71,§7.75.2). EX: ¶82

¶23 e is an **epimorphism** (§7.39) (the dual of a monomorphism) (equiv: is **epic** (Awodey:D2.1)) if

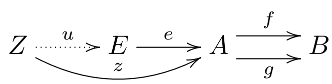
$\forall_{i,j} ie = je \Rightarrow i = j$



If f and g are epis, then so is $g\circ f$; if $g \circ f$ is epi, then so is g. (§7.41) EX: ¶81

¶24 (E, e) is an **equalizer** (§7.51) of f, g iff (UMP) $fe = ge$ and

$\forall_{Z,z.fz=gz}\exists!_u eu = z$

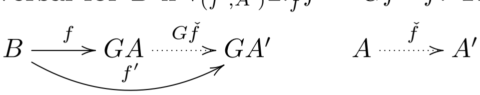


Equalizers are essentially unique (§7.53) and monic (§7.56,§7.59.2). EX: ¶83

¶25 A mono m is a **extremal** (§7.61) if e epic and $m = f \circ e$ implies that e iso.

¶26 Let $G : \mathbf{A} \to \mathbf{B}$ and $B \in \mathbf{B}$. A **G-structured arrow with domain** B is a pair $(f : B \to GA, A)$. (§8.30) It is

- **generating** if $\forall_{r,s:A\to A'} Gr \circ f = Gs \circ f \implies r = s$
- **extremally generating** if it is generating and $\forall_{m:A'\to A, m \text{ mono}, (g,A')} f = Gm \circ g \implies m$ iso.
- **G-universal for** B if $\forall_{(f',A')}\exists!_{\check{f}} f' = G\check{f}\circ f$. That is,

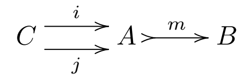


When G is a subcategory inclusion, a G-structured universal arrow is a **reflection** (§4.16).

¶27 $f : A \to B$ is an **isomorphism** if $\exists!_g. f\circ g = id_B \wedge g\circ f = id_A$. (§3.8; ! in §3.11). Every isomorphism is both monic and epic (Awodey:P2.6).

¶28 f is a **monomorphism** (§7.32) (equiv: is **monic** (Awodey:D2.1)) if

$\forall_{i,j} mi = mj \Rightarrow i = j$



If f and g are monos, then so is $g\circ f$; if $g\circ f$ is mono, then so is f. (§7.34) Objects with monomorphisms to X are called **subobjects** of X (Awodey:D5.1). EX: ¶81

¶29 A **point** (Awodey:p32) of C is any $c : 1 \to C$. EX: ¶86

¶30 f is a **regular monomorphism** (§7.56) if it is an equalizer of some pair of morphisms.

¶31 $f : A \to B$ is a **retraction** if $\exists_g. f \circ g = 1_B$ (§7.24) aka **split epi** (Awodey:D2.7). If f and g are retractions, then so is $g \circ f$; if $g \circ f$ is a retraction, then so is g. (§7.27)

¶32 $f : A \to B$ is a **section** if $\exists_g. g \circ f = 1_A$. (§7.19) aka **split mono** (Awodey:D2.7). If f and g are sections, then so is $g \circ f$; if $g \circ f$ is a section, then so is f. (§7.21)

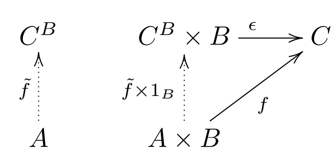
¶33 Several morphism properties combine in useful ways:

- mono, epi $\Rightarrow$ **bimorphism** (§7.49) EX: ¶84
- section $\Rightarrow$ regular mono (§7.35, §7.59.1)
- regular mono $\Rightarrow$ extremal mono (§7.59.2, §7.63)
- retraction $\Rightarrow$ epi (§7.42)
- mono, retraction $\Leftrightarrow$ isomorphism (§7.36)
- section, epi $\Leftrightarrow$ isomorphism (§7.43)

6 Exponentials

¶34 (Awodey:p107,D6.1) In a category with binary products, given two objects B and C, their **exponential** is an object C^B and arrow $\epsilon : C^B \times B \to C$ s.t.

$\forall_{A,f:A\times B\to C}\exists!_{\tilde{f}:A\to C^B}$
$\epsilon \circ (\tilde{f} \times 1_B) = f$



The arrows f and $\tilde{f}$ are "exponential transposes."

¶35 Exponential transposition is self inverse (Awodey:p108). This implies

$$\mathrm{Hom}_{\mathbf{C}}(A \times B, C) \simeq \mathrm{Hom}_{\mathbf{C}}(A, C^B)$$

¶36 The **exponential category** $\mathbf{D}^{\mathbf{C}}$ has as objects functors[¶39] from $\mathbf{C}$ to $\mathbf{D}$ and as morphisms the natural transformations[¶49] between these functors.

¶37 A category is **cartesian closed** (Awodey:p108,D6.2) if it has all finite products and exponentials.

7 Functors

¶38 Default notation here: functors $F, G : \mathbf{A} \to \mathbf{B}$.

¶39 A **covariant functor** (or just **functor**) F (§3.17;Awodey:D1.2) assigns to each **A**-object a **B**-object and to each **A**-morphism a **B**-morphism s.t. composition and identites are *preserved*.

¶40 A **contravariant functor** F (§3.20.5) is a (covariant) functor $\mathbf{A}^{\mathrm{op}} \to \mathbf{B}$.

¶41 A **diagram** (Awodey:D5.15) is a functor $D : J \to C$ from some indexing category J.

¶42 A **endofunctor** has $\mathbf{A} = \mathbf{B}$. $F \circ F$ may be denoted F^2, etc. (§3.23; ftn 15)

¶43 Functors compose. (§3.23)

¶44 A functor $F : C \to D$...

- **preserves limits of type** J if
  $$\forall_{D:J\to C} \forall_{\varprojlim_j D_j} F(\varprojlim_j D_j) \simeq \varprojlim_j F(D_j).$$
- **creates limits of type** J if $\forall_{D:J\to C}$ and all limits $L = \varprojlim_j FD_j$ (i.e., bundle $p_j : L \to FD_j$ in C'), $\exists!(\bar{p_j} : \bar{L} \to D_j) \in C'$ with $F(\bar{L}) = L$, $F(\bar{p_j}) = p_j$, and $\bar{L} = \varprojlim_j D_j$.

¶45 A (covariant) **bifunctor** is a functor from a product category[¶16] (i.e. $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$) such that each partial application is *also* a functor. (See [HHJ12] and bifunctors.tex for more.) A **profunctor** is a bifunctor which is contravariant[¶40] in one argument and covariant in the other; i.e. $\mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{C}$. A **diagonal profunctor** is a profunctor where both elements of the product are the same category; i.e. $\mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{C}$.

¶46 A functor F is (§3.27, §3.33)

- **amnestic** if f is an identity iff Ff is an identity.
- **continuous** if it preserves all limits. (Awodey:D5.24)
- an **equivalence** if it is full, faithful, and isomorphism-dense.
- an **embedding** if it is injective on morphisms.
- **faithful** if $\forall_{A,A'} F|_{\mathbf{A}(A,A')} \subseteq \mathbf{B}(FA, FA')$ is injective.
- **full** if $\forall_{A,A'} F|_{\mathbf{A}(A,A')}$ surjective.
- **isomorphism-dense** if $\forall_B \exists_A . F(A) \simeq B$.

¶47 All functors **preserve** (in **A** implies in **B**) isomorphisms (§3.21), sections (§7.22), and retractions (§7.28).

¶48 Some functors **reflect** (in **B** implies in **A**) useful properties:

- Full, faithful functors reflect sections (§7.23) and retractions (§7.29).
- Faithful functors reflect monos (§7.37.2) and epis (§7.44).

7.1 Transformations

¶49 A **natural transformation** $\tau : F \stackrel{\cdot}{\to} G$ assigns each $A \in \mathbf{A}$ to $\tau_A : FA \to GA$ s.t. $\forall_{f:A\to A'} Gf \circ \tau_A = \tau_{A'} \circ Ff$ (§6.1;Awodey:D7.6). That is,

$$\forall_{A,B,f\in C} \quad Gf \circ \tau_A = \tau_B \circ Ff \qquad \begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\tau_B} & GB \end{array}$$

More generally, given any functor from a product category[¶16], we may say that it is natural in the i-th position if, for all ways of fixing the other positions, the resulting partial applications form natural transformations.

¶50 There is special notation for functors (H) applied to natural transformations and vice-versa (§6.3): $H\tau : HF \stackrel{\cdot}{\to} HG$ defined by $(H\tau)_A = H(\tau_A)$ and $\tau H : FH \stackrel{\cdot}{\to} GH$ defined by $(\tau H)_A = \tau_{HA}$.

¶51 A **dinatural transform** $\theta : R \stackrel{\bullet}{\to} S$ between diagonal profunctors[¶45] $R, S : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{C}$ is a A-object-indexed collection of arrows θ where $\forall_{f:A\to A'\in\mathbf{A}} S\ id_A\ f\theta_A \circ R\ f\ id_A = S\ f\ id_{A'} \circ \theta_{A'} \circ R\ id_{A'}\ f : RA'A \to SAA'$.

7.2 Special Functors

¶52 For every category **C** and object $D \in \mathbf{D}$ there is a unique **constant functor** $!_D$ which sends every C to D and every f to 1_D.

¶53 The **covariant representable functor** (Awodey:p44) at $A \in \mathbf{C}$ is defined by $\mathrm{Hom}(A, —) : \mathbf{C} \to \mathbf{Sets}$. These functors are continuous (Awodey:P5.25).

¶54 Representable functors preserve monos. (§7.37.1)

¶55 Pullback defines a functor
$$h^* : (A \stackrel{\alpha}{\to} C) \in \mathbf{C}/C \mapsto (C' \times_C A \stackrel{\alpha'}{\to} C') \in \mathbf{C}/C'$$
where α' is the pullback of α along h. (Awodey:P5.10)

¶56 $\mathrm{Hom}_{\mathbf{C}}(—, —)$ is a diagonal profunctor[¶45] from $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$, assuming that **C** is locally small.

8 Cones and Sources

¶57 A **cone** (Awodey:D5.15) to a diagram $D(J)$ is a collection of arrows $c_j : C \to D_j$ s.t. $\forall_{D_\alpha \in D(J)} c_j = D_\alpha \circ c_i$. (Cones are also natural transformations[¶49] from the constant functor[¶52] to the inclusion functor of the diagram D. [Mil]) (Cones are sources[¶58] subject to commutation diagrams implied by the diagram.)

¶58 A **source** in category **A** indexed by I is a pair $(A, \{f_i : A \to A_i\}_{i\in I})$. This source has domain A and codomain $\{A_i\}_{i\in I}$. (§10.1)

¶59 Given $(A, \{f_i\}_{i\in I})$ and $\{(A_i, \{g_{ij}\}_{j\in J_i})\}_{i\in I}$ all sources, their **composite** is $(A, \{g_{ij} \circ f_i\}_{i\in I, j\in J_i})$. (§10.3)

¶60 A **mono-source** (§10.5) is $(A, \{f_i\})$ s.t. $\forall r, s : B \to A \, [\forall_{i \in I} f_i \circ r = f_i \circ s] \Rightarrow r = s$.

9 Concrete Categories

¶61 For this section, $\mathbf{A}$ is a **concrete category** over $\mathbf{X}$ with **forgetful** functor[¶39] $U : \mathbf{A} \to \mathbf{X}$ faithful[¶46], denoted $(\mathbf{A}, U)$. (§5.1.1)

¶62 When $\mathbf{A} = \mathbf{X}$, $\mathbf{Alg}(U)$ has U-algebras[¶72] as objects and algebra homomorphisms as morphisms.

¶63 If $\mathbf{X}$ is $\mathbf{Set}$, $\mathbf{A}$ is a **construct**. (§5.1.2)

¶64 $(UA \xrightarrow{f} UB) \in \mathbf{X}$ **is an A-morphism** if f has an *unique* U-preimage in $\mathbf{A}$. (§5.3, §6.22)

¶65 A **free object** $A \in \mathbf{A}$ is one with a (U-structured) universal arrow (u, UA) in B. (§8.22+§8.30) ¶26

10 Adjoints and Adjoint Situations

Be sure to see subsection C.3 for examples.

10.1 Joy Approach

¶66 A functor $G : \mathbf{A} \to \mathbf{B}$ is **adjoint** if $\forall_{B \in \mathbf{B}}$ there exists a G-structured universal arrow with domain B. (§18.1) ¶26

¶67 Adjoints compose (§8.5), preserve mono-sources[¶60] (§8.6), and preserve limits[¶13] (§8.9)

¶68 Given adjoint G with $\eta_B : B \to G(A_B)$ the G-structured universal arrow with domain B, $\exists!_F$ such that $FB = A_B$ and $\eta : id_B \dot{\to} G \circ F$ is natural; further, there is a unique, natural $\epsilon : F \circ G \dot{\to} id_A$ with $G\epsilon \circ \eta G = id_G$ and $\epsilon F \circ F\eta = id_F$. (§19.1)

¶69 $(\eta, \epsilon) : F \dashv G : \mathbf{A} \to \mathbf{B}$ is a **adjoint situation** if the above relationships hold. (§19.7)

10.2 Awodey Approach

¶70 An **adjunction** (Awodey:D9.1) of $F : C \to D$ and $G : D \to C$ is a natural transformation[¶49] $\eta : I_C \dot{\to} (G \circ F)$ s.t.

$$\forall_{f : X \to GY} \exists!_{f^\# : FX \to Y} \; f = Gf^\# \circ \eta_X \qquad \begin{array}{c} FX \\ \downarrow{\scriptstyle f^\#} \\ Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & GFX \\ & \searrow^{f} & \downarrow{\scriptstyle Gf^\#} \\ & & GY \end{array}$$

Equivalently (Awodey:D9.7), a natural *isomorphism*

$$\phi : \mathrm{Hom}_D(FC, D) \simeq \mathrm{Hom}_C(C, GD), \quad \eta_X = \phi(1_{FX})$$

10.3 Moving Right Along

¶71 A **monad** (§20.1) on $\mathbf{X}$ is $(T : \mathbf{X} \to \mathbf{X}, \eta : id_{\mathbf{X}} \dot{\to} T, \mu : T^2 \dot{\to} T)$ s.t.

$$\forall_X \quad \begin{array}{ccc} T^3X & \xrightarrow{T(\mu_X)} & T^2X \\ \downarrow{\scriptstyle \mu_{TX}} & & \downarrow{\scriptstyle \mu_X} \\ T^2X & \xrightarrow{\mu_X} & TX \end{array} \qquad \begin{array}{ccccc} TX & \xrightarrow{T(\eta_X)} & T^2X & \xleftarrow{\eta_{TX}} & TX \\ & \searrow_{id_{TX}} & \downarrow{\scriptstyle \mu_X} & \swarrow_{id_{TX}} & \\ & & TX & & \end{array}$$

A Miscellaneous Terminology

¶72 Given an endofunctor[¶42] F on $\mathbf{C}$, a F**-algebra** is a pair of a **carrier** $X \in \mathbf{C}$ and interpretation morphism $h : FX \to X \in \mathbf{C}$. A **algebra homomorphism** is a morphism f such that $f : (X, h) \to (X', h')$ s.t. $f \circ h = h' \circ T(f)$. (§5.37)

¶73 A category is **finitely presented** (Awodey:p75) if it is the free category over a finite graph quotiented by a finite set of equations.

¶74 The **local membership relation** for generalized element $z : Z \to C$ and subobject M (i.e., with monic $m : M \to C$), $z \in_X M$, holds iff $\exists_{f : Z \to M} . z = mf$.

¶75 An ω**-complete Partial Order** (ωCPO) is a Poset which has all *co*limits of type $(\mathbb{N}, \leq)$. (All countably infinite ascending chains have a top.) (Awodey:p101,E5.33)

B Miscellaneous Useful Properties

¶76 (Awodey:p84,L5.8) In the commuting diagram

$$\begin{array}{ccccc} F & \xrightarrow{f'} & E & \xrightarrow{g'} & D \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

1. If $FEBA$ and $EDCB$ are pullbacks, so is $FDCA$.
2. If $FDCA$ and $EDCB$ are pullbacks, so is $FEBA$.

¶77 (Awodey:p84,C5.9) Pullbacks preserve commutative triangles.

¶78 (?) Monic arrows pullback to monic arrows. (In the above, if g is monic, so is g'.)

¶79 Universal Constructions (or Universal Mapping Properties, UMP) reduce to limits (Awodey:p91,E5.17-20):

terminals	products	equalizers	pullbacks
	$x \quad y$	$x \overset{\alpha}{\underset{\beta}{\rightrightarrows}} y$	$\begin{array}{ccc} & & x \\ & & \downarrow \\ y & \to & z \end{array}$

¶80 Objects defined by UCs are unique up to isomorphism.

C Examples To Jog Your Memory

C.1 Set

¶81 Epic[¶23] is surjective, monic[¶28] is injective.

¶82 Coequalizers[¶22] correspond to equivalence classes (§7.69.1): Let $\sim$ be *the smallest* eq. rel. s.t. $\forall_{a \in A} f(a) \sim g(a)$; then $(Q, q) = (B/\sim, b \mapsto [b]_\sim)$ is a coequalizer of f and g.

¶83 Equalizers[¶24]: $(E, e) = (\{x \mid f(x) = g(x)\} \subseteq X, \subseteq)$.

C.2 Mon

¶84 Bimorphisms[¶33] are not isos: $((\mathbf{N}, +, 0) \to (\mathbf{Z}, +, 0))$. (Pierce:§1.6.3)

¶85 $(\{*\}, \cdot, *)$ is a (the) zero[¶21].

¶86 Each monoid M has only one point[¶29], $1 \to M$.

C.3 Adjoint Situations and Monads

Defintitons in section 10.

¶87 Consider $(\eta, \epsilon) : F \dashv G : \mathbf{Mon} \to \mathbf{Set}$. $\eta_X : X \to GFX$ is insertion of generators: $\forall x \in X \eta_X x = x$. $\epsilon_Y : FGY \to Y$ is the re-introduction of structure; if $FGY = ((GY)^*, \cdot, \varepsilon)$ and $Y = (GY, +, 0)$ then

$$\epsilon_Y \varepsilon = 0 \quad \epsilon_Y(y \cdot z) = y + z \quad \epsilon_Y(y \in GY) = y$$

¶88 Further, $T = G \circ F$ is a monad. Generically, μ...

$$\mu_X(TTX) = (G\epsilon F)_X(TTX) = (G\epsilon_{FX})(GFGFX)$$
$$= G((\epsilon_{FX})(FGFX)) = GFX$$

So here μ is the G-image of a function which takes $y \in FGFX = F(X^*)$ (that is, a concatenation of symbols from GFX) and re-imposes structure to obtain $\epsilon_{FX} y \in FX$.

D Bootstrapping Category Theory

¶89 **Cat** is the category which has locally small categories as objects and functors[¶39] as morphisms. (It is not, itself, locally small, and so is not an object in itself.) **Cat** is cartesian closed[¶37] (see product category[¶16] and exponential category[¶36]). Its initial object is the empty category and its terminal object is the category of a single object and its identity morphism.

Index

References

[Awo10] Steve Awodey. *Category Theory*. Oxford University Press, Oxford; New York, August 2010.

[HHJ12] Ralf Hinze, Jennifer Hackett, and Daniel W. H. James. Functional Pearl: F for Functor. *ICFP*, 2012. URL: http://www.cs.ox.ac.uk/people/daniel.james/functor/functor.pdf.

[Hin] Ralf Hinze. Kan extensions for program optimisation or: Art and dan explain an old trick. URL: http://www.cs.ox.ac.uk/ralf.hinze/Kan.pdf.

[JHS04] Jirı Adámek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. 2004. URL: http://www.prgmea.com/pdf/abstract/9.pdf.

[Mil] Bartosz Milewski. Understanding limits. URL: http://bartoszmilewski.com/2014/05/08/understanding-limits-2/.

[Pie91] Benjamin C. Pierce. *Basic Category Theory for Computer Scientists*. Foundations of Computing Series. MIT Press, Cambridge, Massachusetts, August 1991.

[Rie] Emily Riehl. *Category Theory in Context*. URL: http://www.math.jhu.edu/~eriehl/727/context.pdf.