$\Sigma \dashv \Delta \dashv \Pi$

June 27, 2018

1 Getting Started

The "diagonal functor" $\Delta : \mathcal{C} \to \mathcal{C}^2$ is charmingly degenerate: $\Delta X = (X, X), \Delta(f : A \to B) = (f, f) : (A, A) \to (B, B)$. (In the ^{2} category, morphisms are such that $(f, g)(a, b) = (fa, gb)$.) For notational clarity, we'll use $A \times B$ for products within $\mathcal{C}$ and (A, B) for objects in $\mathcal{C}^2$.

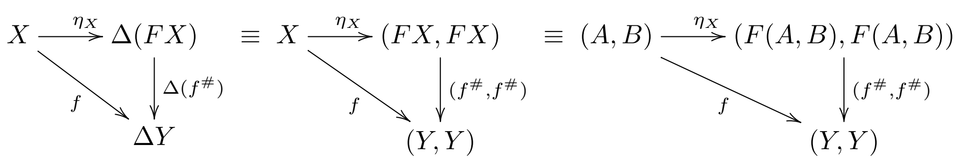
Define $\Pi_1 : \mathcal{C}^2 \to \mathcal{C}$ to be the first projector, and Π_2 the second. If $\mathcal{C}$ has products, then we may define $\Pi : \mathcal{C}^2 \to \mathcal{C}$ as $(A, B) \mapsto A \times B$, and $\Pi_1 = \pi_1 \circ \Pi$. If $\mathcal{C}$ has coproducts, define $\Sigma : \mathcal{C}^2 \to \mathcal{C}$ by $(A, B) \mapsto A + B$.

While on the topic of notation, recall that, if $\mathcal{C}$ is so equipped, we may form arrows involving products and coproducts from others: $\langle f : A \to B, g : A \to C \rangle (a) = (fa) \times (ga) \in B \times C$ and $[f : A \to C, g : B \to C] ((a : A) + (b : B)) \in C$ is case analysis.

2 Left Adjoint

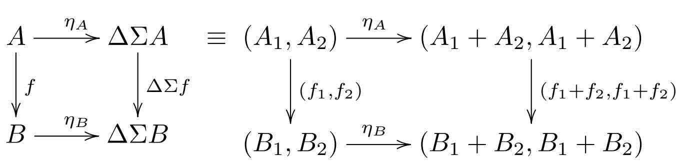
2.1 Unit

What would a left adjunction to Δ be? It would be a functor $F : \mathcal{C}^2 \to \mathcal{C}$ and natural transformation η where, in the category $\mathcal{C}^2$,



If this diagram is to commute for all f, then: $\Pi_1 \circ f = \Pi_1 \circ (f^\#, f^\#) \circ \eta_X = f^\# \circ \Pi_1 \circ \eta_X$, and similarly for the Π_2 component. Intuitively, this can only work in the case where $f^\#$ is able to discriminate whether it has been handed the Π_1 or Π_2 projection of η_X's output. That sounds like a perfect use of coproducts! If we take $\eta_X = (i_1, i_2) : X \to \Delta(\Sigma X)$ (i.e. $\eta_X(x) = (i_1 x, i_2 x) : (A, B) \to (A + B, A + B)$) and define $f^\# = [\Pi_1 \circ f, \Pi_2 \circ f]$, then we see that $f^\# \circ \Pi_1 \circ \eta_X = [\Pi_1 \circ f, \Pi_2 \circ f] \circ \Pi_1 \circ (i_1, i_2) = [\Pi_1 \circ f, \Pi_2 \circ f] \circ i_1 = \Pi_1 \circ f$ as required. Any such $f^\#$ is clearly unique.

All that remains is to check that η_X is natural from I to $\Delta\Sigma$. That is, does this commute for all $f : A \to B$?



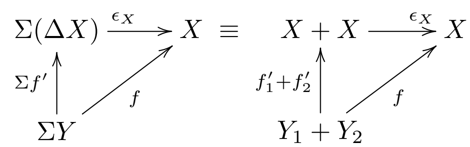
Well:

$$\begin{aligned}
\Delta\Sigma f \circ \eta_A &= ((f_1 + f_2), (f_1 + f_2)) \circ (i_1, i_2) && \text{defn } \eta, \Delta, \Sigma \\
&= ((f_1 + f_2) \circ i_1, (f_1 + f_2) \circ i_2) && \circ \\
&= (i_1 \circ f_1, i_2 \circ f_2) && (f + g) \circ i_1 = i_1 \circ f \\
&= (i_1, i_2) \circ (f_1, f_2) && \circ \\
&= \eta_B \circ f && \text{defn } \eta, f
\end{aligned}$$

So we have: $\Sigma \dashv \Delta$.

2.2 Counit

Looking at this the other way, we have, in $\mathcal{C}$,

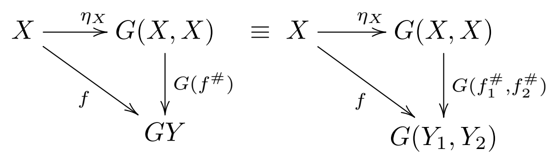


Then if we take $\epsilon_X = [id, id]$ we can define $f' = (f \circ i_1) + (f \circ i_2)$. This is unique and ϵ_X is natural by inspection.

3 Right Adjoint

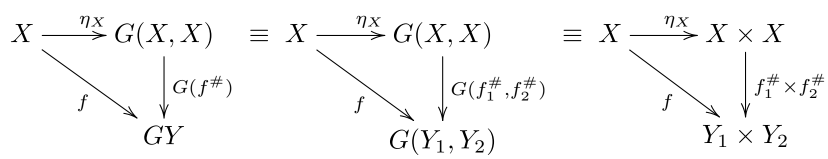
3.1 Unit

What about the other way around? Now we have, in $\mathcal{C}$ this time,



Let's speculate that $G = \Pi_1$ and see what goes wrong. That would mean that, for each $f : X \to Y$, there is some unique $f^\# : (X, X) \to (Y, Y')$ such that $f = \Pi_1 f^\# \circ \eta_X$. But that can't possibly be true, because given such a $f^\#$, one that differs only in its second component will also work, so we've violated the "exists unique" part of the definition.

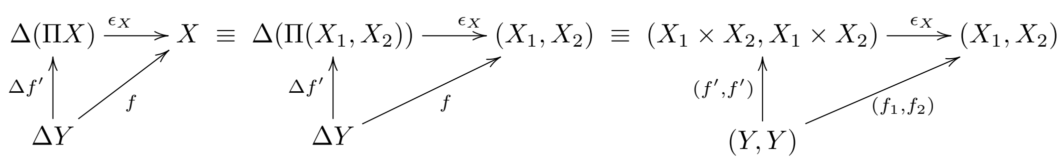
But if we take $G = \Pi$, then



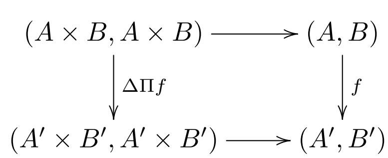
And we can see that taking $\eta_X = \langle id, id\rangle$ and $f^\# = (\pi_1 \circ f) \times (\pi_2 \circ f)$ makes this commute with a unique $f^\#$ for each f. η_X is clearly natural. Thus we have that $\Delta \dashv \Pi$.

3.2 Counit

Here the counit diagram takes place in $\mathcal{C}^2$:



Take $\epsilon_X = (\pi_1, \pi_2)$, then $f' = \langle f_1, f_2\rangle$. Uniqueness of f' is immediate. Naturality of ϵ_X is immediate from the action of $\Delta\Pi$ on arrows:



$\Delta\Pi f = \Delta\Pi(f_1, f_2) = \Delta(f_1 \times f_2) = (f_1 \times f_2, f_1 \times f_2)$, and so $(\pi_1, \pi_2) \circ \Delta\Pi f = (\pi_1, \pi_2) \circ (f_1 \times f_2, f_1 \times f_2) = (f_1, f_2) = (f_1, f_2) \circ (\pi_1, \pi_2)$.

4 Notes

Note that for the unit of the left adjunction and the counit of the right adjunction, we had to choose "non-obvious" natural transformations, whereas for the other two we had things "built from identities". In the former two cases, there are actually other functions which would work, notably $\eta_X = \langle i_2, i_1\rangle$ and $\epsilon_X = (\pi_2, \pi_1)$.

$\Delta \dashv \Pi$ has some reading as "diagonals are free products" though I do not find that terribly informative; I have yet to find "coproducts are free diagonals" a useful statement at all.